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An Integral Equation Method for the Numerical Solution of a Two-Dimensional Vertical Jet under Gravity

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A method for the numerical solution of the singular integral equations related to a vertical jet problem is developed. A jet flow from an aperture on the bottom of a large vessel fully filled with liquid, under gravity, has been studied. The problem has been programmed and run on a computer, and the computed results are shown.

1. FORMULATION OF THE PROBLEM

Consider a large rectangular vessel (see Fig. 1a) filled with water, with an aperture located at the center of its bottom. A jet issues downward from the aperture under gravity, with two free surfaces. Assume that the height of the vessel is h_1 and the half-width of the aperture b. The flow is assumed to be steady, two dimensional, inviscid, and incompressible, and the motion irrotational. The fluid motion is symmetric with respect to the center line BC_∞, and hence only half of the fluid region has to be considered (see Fig. 1b). It has been shown by Carter [1] that the free streamlines should be asymptotic to this vertical line BC_∞. From the conservation of mass, the velocity far downstream from the aperture approaches infinity. We choose the origin of the xy-plane at the point E, the x-axis from left to right and the y-axis upward. Assume that the water surface on the top AB of the vessel is horizontal, and the velocity on it constant and equal to q_1. Let x_1 be the half-width of the vessel. Furthermore, we assume the velocity on DC is constant and equal to q_2, where D is a point some distance from the origin E and DC is a horizontal line. Let the length of DC be x_2.

Let ϕ be the velocity potential and ψ the stream function. Let $W = \phi + i\psi$ and $z = x + iy$. Then $dW/dz = qe^{-i\theta} = u - iv$. Now we map the fluid region ABCDEF in the z-plane onto the rectangle ABCDEF in the W-plane (see Figs. 1b and 2).

We introduce dimensionless variables

$$\overline{W} = \frac{W}{\psi_1}, \qquad \bar{q} = \frac{q}{q_0}, \qquad \bar{g} = \frac{g\psi_1}{q_0^{\,3}}, \tag{1}$$

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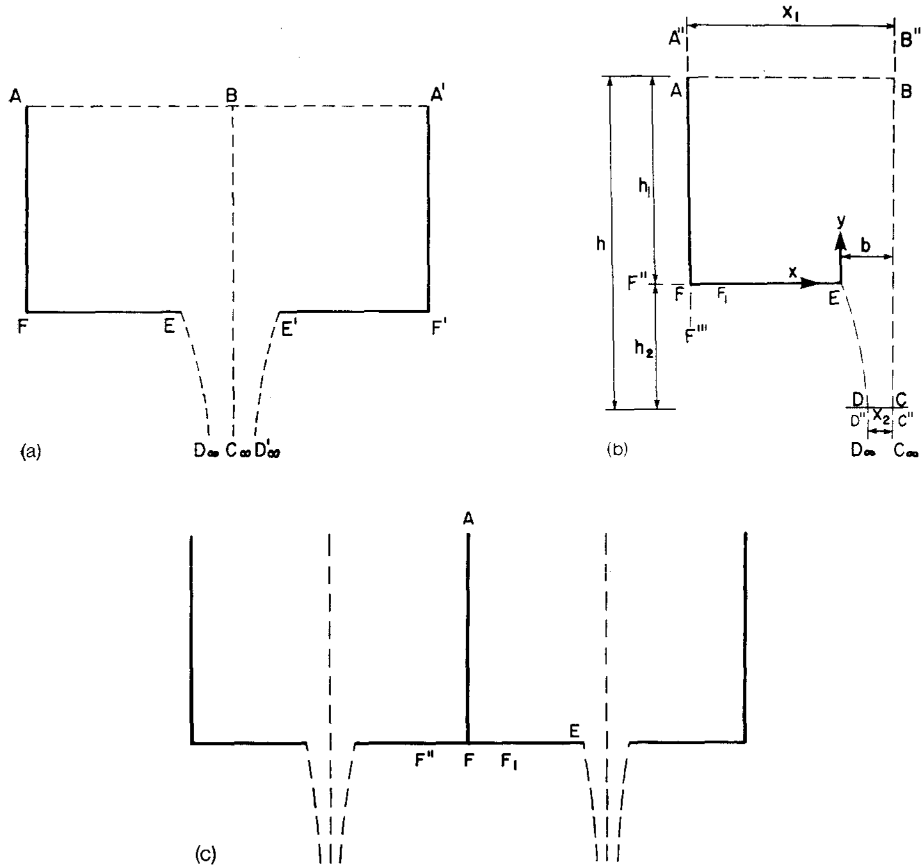

FIG. 1a. The physical plane for a 2-D vertical jet from a vessel.
FIG. 1b. Half of the physical plane for a 2-D vertical jet from a vessel.
FIG. 1c. Multiple jets from vessels.

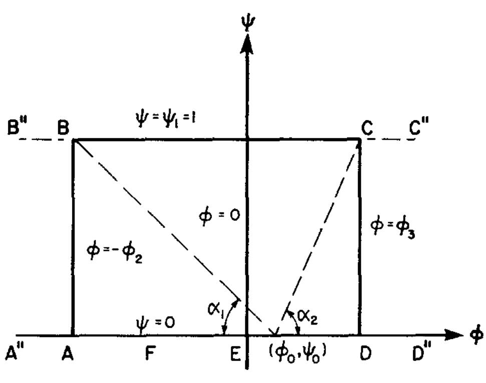

FIG. 2. The W-plane for a 2-D vertical jet from a vessel.

and then drop all the bars, for convenience. All variables hereafter are *dimensionless*. Note that q_0 is the fluid speed at the origin E. The integral form of the free surface condition in these dimensionless variables is given by

$$q^3 + 3g \int_0^{\phi} \sin \theta(t)\, dt = q_0^{\,3} = 1, \tag{2}$$

where, in the dimensionless form, $q_0 = 1$ and $\psi_1 = 1$.

If all q's on four sides of the rectangle ABCDEF, and θ on the surface ED, have been computed, then the form of the free surface (x, y), the heights h, h_1, and h_2 (see Fig. 1b), the half-width b of the aperture, the half-width x_1 of the vessel, the length x_2 of DC, and other quainities may be computed. Here ϕ_2, ϕ_3 (and hence $\phi_1 = \phi_2 + \phi_3$), q_1, the velocity on AB, and q_2, the velocity on DC, are assumed to be given. From the conservation of mass, we have $q_1 x_1 = q_2 x_2 = \psi_1 = 1$. The two equations for computing x and y on the surface are

$$\begin{aligned} x(\phi) &= \int_0^{\phi} \frac{\cos \theta(t)\, dt}{q} \qquad \text{along ED,} \\ y(\phi) &= \int_0^{\phi} \frac{\sin \theta(t)\, dt}{q} \qquad \text{along ED.} \end{aligned} \tag{3}$$

Thus

$$\begin{aligned} b &= x_{\mathrm{D}} + x_2 , \\ h_2 &= -y_{\mathrm{D}} , \qquad \text{since} \quad x_{\mathrm{E}} = y_{\mathrm{E}} = 0. \end{aligned} \tag{4}$$

The height h (see Fig. 1b) can be computed from

$$\begin{aligned} h &= \int_{-h_2}^{h_1} dy \\ &= \int_0^{\phi_1} \frac{1}{q(\phi)}\, d\phi \qquad \text{along BC,} \end{aligned} \tag{5}$$

and hence

$$h_1 = h - h_2 . \tag{6}$$

2. Some Other Solutions

Helmholtz (1868) and Kirchhoff (1869) investigate this problem for $g = 0$ (for the sources, see Lamb [4, pp. 75, 96]). They express the shape of the jet in terms of the half-width b of the aperture. They calculate the final breadth of the stream, between the free surfaces, in terms of b, as $2\pi b/(2 + \pi)$, and hence the coefficient of contraction is $\pi/(2 + \pi) = 0.611$. For further discussion, see Lamb [4, pp. 94–99].

In Section 1 we mentioned that our solution will be formulated on the assumption that ϕ_2, ϕ_3, q_1, and q_2 are prescribed. This is an inverse problem. On the other hand, we might be interested in the solution when x_1, h_1, and b are the given quantities. This can be done by applying the successive relaxation method to the Laplace equation $\partial^1\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0$. The method will involve subdividing the whole fluid region and relocating the free surfaces. For more details, see Southwell and Vaisey [6].

3. Cauchy Integral Equations

Let Γ be a simple closed contour, taken in the positive sense (counterclockwise), such that the function $f(z)$ is analytic at every point on and inside Γ. Then the Cauchy integral formula is

$$\pi i f(z_0) = \int_\Gamma \frac{f(z)\,dz}{z - z_0}, \tag{7}$$

where z_0 is a point on Γ at which the slope of the tangent to Γ is continuous.

Let $f(z) = U(x, y) + iV(x, y)$ and $z - z_0 = \rho e^{i\alpha}$. Let s be the arc length along the contour Γ and $\mathbf{n}$ the inward unit normal to Γ. Then (7), after the real and imaginary parts are separated and integration by parts and the Cauchy–Riemann conditions are applied, becomes, with some simplifications,

$$\begin{aligned} \pi U(x_0, y_0) &= -\int_\Gamma \frac{\partial V(x, y)}{\partial s} \ln \rho \, ds + \int_\Gamma U(x, y) \frac{\partial \alpha}{\partial s} \, ds, \\ \pi V(x_0, y_0) &= \int_\Gamma \frac{\partial U(x, y)}{\partial s} \ln \rho \, ds + \int_\Gamma V(x, y) \frac{\partial \alpha}{\partial s} \, ds. \end{aligned} \tag{8}$$

Equations (8) are of the same form as the equation of Jaswon [3, Eq. (18)] known as Green's boundary formula.

4. Numerical Solution

Let Γ be the rectangle ABCDEF in the W-plane in Fig. 2, and let ϕ_2, ϕ_3 (and hence $\phi_1 = \phi_2 + \phi_3$), q_1, and q_2 be known quantities. Divide each horizontal side of the rectangle into a number N (N is an even integer) of subintervals. Let h_x be the length of each subinterval, and $x^{(0)} = -\phi_2$, $x^{(1)} = -\phi_2 + h_x, \ldots$, $x^{(j)} = -\phi_2 + jh_x, \ldots$, $x^{(N)} = \phi_3$ the nodal points. Then (8) can be written as the discrete sums

$$\begin{aligned}
\pi U(x_0, y_0) &= -\sum_j \int_j \frac{\partial V(x, y)}{\partial s} \ln \rho \, ds + \sum_j \int_j U(x, y) \frac{\partial \alpha}{\partial s} \, ds, \\
\pi V(x_0, y_0) &= \sum_j \int_j \frac{\partial U(x, y)}{\partial s} \ln \rho \, ds + \sum_j \int_j V(x, y) \frac{\partial \alpha}{\partial s} \, ds + \alpha_1 q_1 + \alpha_2 q_2,
\end{aligned} \tag{9}$$

where (x_0, y_0) $(=(\phi_0, \psi_0))$ is a noncorner point on BC, AF, FE, or ED, α_1 and α_2 are angles shown in Fig. 2, and the integrals in (9) are ranged over pairs of subintervals (that is, $j = 1, 3, 5, \ldots, (N - 1)$). We approximate each integral, except for those which contain a logarithmic singularity (explained later), by Simpson's formula, that is,

$$\begin{aligned}
\int_j f(x)\, dx &= \int_{x^{(j)} - h_x}^{x^{(j)} + h_x} f(x)\, dx \\
&= \frac{h_x}{3} [f(x^{(j)} - h_x) + 4f(x^{(j)}) + f(x^{(j)} + h_x)] + O(h_x^5),
\end{aligned}$$

where $f(x)$ is any integrand of (9).

Symm [5] assumes that U, V, $\partial U/\partial s$, $\partial V/\partial s$ are constants in each subinterval and approximates the integrals with remaining integrand. Hence the results Symm obtains are exact when U and V are constants; that is, it is a first-order approximation. Recall that Symm applies the numerical technique on equations derived by Jaswon [3], which are of the same form as (8).

From now on, we write

$$\begin{aligned}
U &= U(\phi, \psi) = u, \\
V &= V(\phi, \psi) = -v,
\end{aligned} \tag{10}$$

where u and v are components of the fluid velocity. Note that $\theta = 0$ on EF; $\theta = -\pi/2$ on AB, CD, FA, and BC, and $q = q_1$ and $q = q_2$ on AB and CD, respectively. In terms of U and V, $U = 0$ on AB, BC, CD, and FA, and $V = 0$, q_1 and q_2 on EF, AB, and CD, respectively.

One particular difficulty which appears in the computing procedure is the logarithmic singularity at $W_0 = \phi_0 + i\psi_0$ in (9) when the interval contains W_0. This is overcome by developing a formula using Maclaurin's series and term-wise integration. The formula we obtain is

$$\begin{aligned}
&\int_{\phi_0 - h_x}^{\phi_0 + h_x} \frac{\partial U(\phi, \psi)}{\partial \phi} \ln |\phi - \phi_0| \, d\phi \\
&\quad = 2h_x(\ln h_x - 1)\left(\frac{\partial U}{\partial \phi}\right)_0 + \frac{h_x^3}{9}(3 \ln h_x - 1)\left(\frac{\partial^3 U}{\partial \phi^3}\right)_0 + O(h_x^5(5 \ln h_x - 1)),
\end{aligned} \tag{11}$$

where the subscript 0 refers to the point (ϕ_0, ψ_0). The same formula can be used for

$$\int_{\phi_0-h_x}^{\phi_0+h_x} \frac{\partial V(\phi, \psi)}{\partial \phi} \ln | \phi - \phi_0 | \, d\phi.$$

The derivatives $(\partial U/\partial \phi)_0$ and $(\partial^3 U/\partial \phi)_0$ are easily obtained using Taylor series expansions,

$$60h_x \left(\frac{\partial U}{\partial \phi}\right)_0 = 45(U_1 - U_{-1}) - 9(U_2 - U_{-2}) + (U_3 - U_{-3}) + O(h_x{}^7),$$

$$8h_x{}^3 \left(\frac{\partial^3 U}{\partial \phi^3}\right)_0 = -13(U_1 - U_{-1}) + 8(U_2 - U_{-2}) - (U_3 - U_{-3}) + O(h_x{}^7), \quad (12)$$

$$60h_x \left(\frac{\partial U}{\partial \phi}\right)_0 = -147U_0 + 360U_1 - 450U_2 + 400U_3 - 225U_4 + 72U_5 - 10U_6 + O(h_x{}^7),$$

$$8h_x{}^3 \left(\frac{\partial^3 U}{\partial \phi^3}\right)_0 = -49U_0 + 232U_1 - 461U_2 + 496U_3 - 307U_4 + 104U_5 - 15U_6 + O(h_x{}^7), \quad (13)$$

$$60h_x \left(\frac{\partial U}{\partial \phi}\right)_0 = -10U_{-1} - 77U_0 + 150U_1 - 100U_2 + 50U_3 - 15U_4 + 2U_5 + O(h_x{}^7),$$

$$8h_x{}^3 \left(\frac{\partial^3 U}{\partial \phi^3}\right)_0 = -15U_{-1} + 56U_0 - 83U_1 + 64U_2 - 29U_3 + 8U_4 - U_5 + O(h_x{}^7), \quad (14)$$

and

$$60h_x \left(\frac{\partial U}{\partial \phi}\right)_0 = 2U_{-2} - 24U_{-1} - 35U_0 + 80U_1 - 30U_2 + 8U_3 - U_4 + O(h_x{}^7),$$

$$8h_x{}^3 \left(\frac{\partial^3 U}{\partial \phi^3}\right)_0 = -U_{-2} - 8U_{-1} + 35U_0 - 48U_1 + 29U_2 - 8U_3 + U_4 + O(h_x{}^7), \quad (15)$$

where $U_j = U_j(\phi_0 + jh_x, \psi_0)$.

The same formulas can be used for $(\partial V/\partial\phi)_0$ and $(\partial^3 V/\partial\phi^3)_0$. Note that formulas (12) are symmetric; (13), (14), and (15) are nonsymmetric; and (13) gives one-sided derivatives.

We compute the derivatives of U and V (for a point on one side of the rectangle ABCDEF) using (12), or (13)–(15). Here we use (12) if we know all the necessary values of U or V (for a corner or points near a corner, this means we know the values of U or V outside the fluid region); otherwise (13)–(15) should be applied. When we use (12) for a corner point or points near a corner, we may need the values of V on AA″, BB″, CC″ and DD″, and U on DD″ (see Fig. 2), and U on FF″ and V on FF‴ (see Figs. 1a and b). However, we do not know V on AA″, BB″, and FF‴ (and hence we cannot use (12) here). For V on CC″ and DD″, and U on DD″, we use the Lagrangian cubic-interpolation formula (see Fröberg [2]). For U on FF″, we might consider we are dealing with multiple jets (see Fig. 1c), so that U on FF″ and the corresponding U on FF_1 have the same magnitudes but different signs.

We estimate the initial values of U and V on all sides of the rectangle ABCDEF and start the Gauss–Seidel iterative procedure. We compute new values of V on BC, FA, and ED, and U on FE, using (9) (together with (11); note that (11) is applied for the case $\phi_0 = (2j - 1)\, h_x$, $j = 1, 2, \ldots, N/2$, we use $2h_x$ instead of h_x in (11) when $\phi_0 = 2jh_x$, $j = 1, 2, \ldots, N/2 - 1$), and evaluate all necessary derivatives of U and V using (12), or (13)–(15). From the free surface condition (2), we obtain a new (dimensionless) g, since in this case we take $q = q_2$ and $\phi = \phi_3$ in (2) (that is, we compute g using $g = (1 - q_2^3)/(3 \int_0^{\phi_3} \sin(\theta(t))\, dt)$). Knowing g, we compute U on ED also, using (2), and evaluate the necessary derivatives of U using (12). We repeat the procedure until the successive approximations differ by a prescribed small number 10^{-k}, e.g., $k = 6$.

TABLE I

$q_1 = 0.025$, $q_2 = 1.11111$

ϕ_1	h_1	h_2	b	q_{O+h}	g
3.8	29.19	0.86	1.45	0.047	0.721
4.1	37.46	0.94	1.47	0.039	0.660
4.4	48.66	1.02	1.50	0.031	0.608

TABLE II

$q_2 = 1.11111$

q_1	x_1	ϕ_1	h_1	h_2	b	q_{O+h}	g
0.025	40.	4.4	48.66	1.017	1.50	0.031	0.608
0.0125	80.	4.5	67.074	1.044	1.502	0.02	0.592
0.01	100.	4.6	80.61	1.07	1.51	0.016	0.577
0.008	125.	4.65	91.56	1.085	1.512	0.013	0.570

TABLE III

$q_2 = 1.15$

q_1	x_1	ϕ_1	h_1	h_2	b	q_{O+h}	g
0.025	40	4.54	49.81	1.044	1.4787	0.032	0.6172
0.0125	80.	4.65	65.00	1.060	1.4827	0.022	0.608
0.01	100.	4.7	77.78	1.089	1.4897	0.018	0.591
0.008	125.	4.75	86.364	1.100	1.4923	0.015	0.584

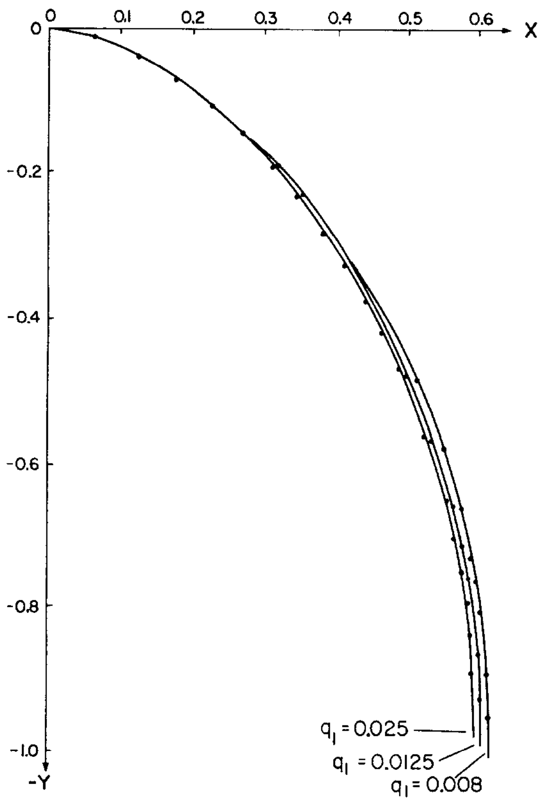

FIG. 3. Shapes of jet near aperture E (when $q_2 = 1.11111$).

5. Numerical Results and Discussions

It was mentioned above that we normalized the flux $\psi_1 = 1$ and the fluid speed at the origin E, $q_0 = 1$. We may expect the iterative procedure outlined above to work well for ϕ_1 not too large (say $\phi_1 < 10$), and q_1 not close to 1 nor too small (say $0.0005 < q_1 < 0.026$). When ϕ_1 is large or q_1 small, we have the jet flow with vessel height h_1 large; and if q_1 approaches 1, vessel height $h_1 \to 0$, hence we may not assume the velocity on the surface AB of the vessel to be constant.

We fix ϕ_1 and q_1, and then compute h_1, h_2, h, b, x_1, x_2, x_2/b, and q_{O+h}, where q_{O+h} is the fluid speed at the very first node near B on BC (see Figs. 1 and 2). For $q_1 = 0.025$ and $q_2 = 1.11111$, Table I shows how ϕ_1 affects the solution. From the computed results, it shows that the vessel height h_1 and the half-width of the aperture b increase as ϕ_1 increases. For $q_2 = 1.11111$ and $q_2 = 1.15$, Tables II and III, respectively, show four different values of q_1 with suitably chosen values of ϕ_1, and the corresponding values of other parameters.

We use 29 points on each horizontal side of the rectangle and, in most cases, it requires only eight cycles to obtain up to three decimal places, and twenty cycles for six places.

When successive approximations (of U and V) agree up to the desired number of decimal places, we compute the shape (x, y) of the free surface ED using (3). Figures 3 and 4 show some such free surfaces.

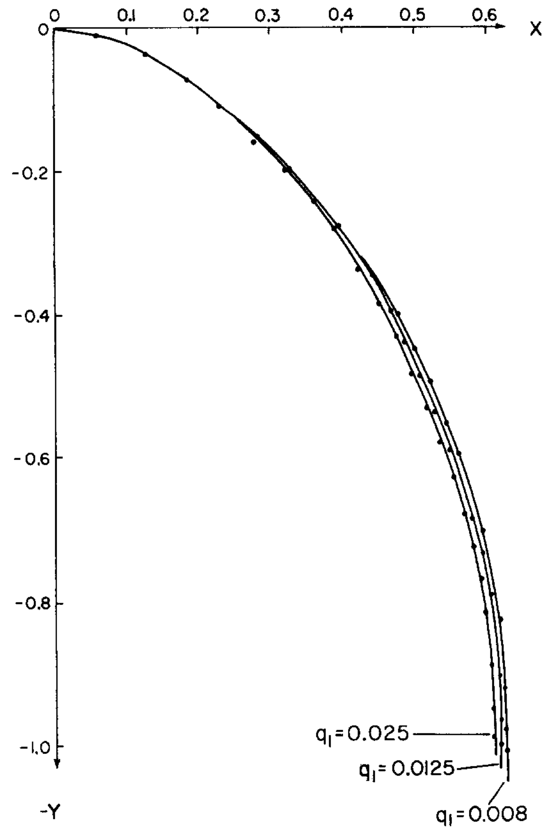

FIG. 4. Shapes of jet near aperture E (when $q_2 = 1.15$).

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